Single server queues with several services

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Abstract

A service station offers several services of which one is the required and the others are harmful (or destructive) for each customer. At the time when selected for service customers enter in correct mode of service according to a Bernoulli process with parameter p which is the probability of being selected in correct mode. Arrival follows Markovian arrival process and service time is Phase type distributed in both undesired and required phases. An exponentially distributed threshold clock starts ticking if a customer enters to incorrect mode and the service is terminated if the clock realizes before the customer is transferred to correct service mode. The rate of loss of customers, rate of customers leaving with correct service starting with incorrect service are computed.

Keywords: desired/undesired service states, random threshold clock, Markovian arrival process

1 Introduction

In the analysis of classic queueing models it is assumed that the server is completely aware of the exact service requirement of a customer (see Gross and Harris [2]). Quite often only one type of service is offered by the system and so conflict does not occur. It is also true that the customer

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knows the type of service he needs. Thus there is no conflict on the service provided to the customer. However, there are several real life situations where the server and/customer are (is) not knowledgeable about the exact service requirement. This is especially the case when several types of services are available at a service station. As concrete example we have vehicles for repair at service stations, patients consulting physicians for diagnosis and medication. If the right service required is not identified and instead the diagnosis turned out to be wrong the result could be disastrous. A wrong diagnosis and consequent service provided may sometimes turn out to be even fatal/ may result in the equipment getting service, rendered totally unusable. It is this type of problem that we analyze in this note.

In real life there are several service providing system offering a multitude of service. Neither the server nor the customer may be fully aware of the exact service requirement. This, very often, results in irreperable damage to the customer being served. We study such a system in this paper with each customer requiring a specific service. However, due to wrong diagnosis service does not start necessarily for the required one.

The article is organized as follows. In section 2, the mathematical model is described. This section also provides the steady state analysis and some performance measures. Several cases of the above model are considered in Sections 3 and 4. An illustration is given in Section 5. Numerical illustrative example is described in Section 6. In Section 7 we extend the above results to the case of arbitrarily distributed service time in the undesired and desired stages of service. However, we restrict $n_1 = n_2 = 1$. Further we relabel the undesired service as preliminary and desired as main services to suit certain context of application.

List of notations and abbreviations used:

$CTMC$: Continuous time Markov chain
$LIQBD$: Level independent quasi-birth and death process
MAP: Markovian arrival process
LST: Laplace Stieltjes Transform
e: Column vector of $1'$s with appropriate order

2 Model Description

The assumptions leading to the formulation of the mathematical model are

• An infinite capacity queueing system where a single server is providing both unwanted (incorrect) and required (correct) services.

• Arrival of customers to the system is according to the MAP (Markovian arrival process). In a MAP, the customers arrival is directed by an irreducible $CTMC$ (continuous time Markov chain) $\{\phi_t, t \geq 0\}$ with the state space $\{1,2,\ldots,m\}$. The transition intensities of the Markov chain $\{\phi_t, t \geq 0\}$ which are accompanied by arrival of k customers are described by the matrices $D_k, k = 0,1$. Vector $\boldsymbol{\eta}$ of the stationary distribution of the process $\{\phi_t, t \geq 0\}$ is the unique solution to the system

$$\boldsymbol{\eta}(D_0 + D_1) = \boldsymbol{\eta}D = 0 and \boldsymbol{\eta}e = 1. \tag{1}$$

Fundamental rate λ of the MAP is given by $\lambda = \boldsymbol{\eta}D_1 e$.

• A customer is selected for desired (required) service with probability p or to the incorrect service with probability $q = 1 - p$.

• (β_1, S_1) of order n_1 gives the PH-representation for the duration of the correct service time distribution when the service of a customer starts in correct service mode. Let ${}^{0}_{1}$ be such that $S_1 e + \quad S_1^0 = 0$. Let $\mu'_1 = \beta_1(-S_1)^{-1}e$ be the mean of this PH-representation.

• (β_2, S_2) of order n_2 gives the PH-representation for the duration of the incorrect service time distribution when the service of a customer starts in incorrect service mode. The rate (vector) of loss is given by $_2^0$ and the rate (vector) of getting into correct service mode is given by S_2^0 . Note that $S_2e + \ S_2^0 + \ S_2^0 = 0$. Let $\mu'_2 = \beta_2(-S_2)^{-1}e$ be the mean of this PH-representation.

• (β_3, S_3) of order n_3 gives the PH-representation for the duration of the correct service time distribution when the customer has gone through incorrect service initially. Let $_3^0$ be such that $S_3e + \ S_3^0 = 0$. Let $\mu'_3 = \beta_3(-S_3)^{-1}e$ be the mean of this PH-representation. [**NOTE**: One can take this to be same as (β_1, S_1) but it looks more meaningful in most applications that the service time after going through incorrect one to be different from the use of directly getting into required service. Just something to keep in mind.]

• The service time of a customer can be modeled as a PH-distribution with representation (β, S) of order $n = n_1 + n_2 + n_3$, where

$$\beta = (p\,\beta_1, q\,\beta_{2,0}) \tag{2}$$

$$S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & S_2^0\ \beta_3 \\ 0 & 0 & S_3 \end{pmatrix} \tag{3}$$

Let ^{0} be such that $Se + \ S^0 = 0$ and S^0 is given by $^0 = [\ _1^0 \quad _2^0 \quad _3^0 \quad]^T$.

Let $N(t)$ be the number of customers in the system, $N^*(t)$ the mode of service going on whether direct admission to required/ undesired or one that came from undesired service designated by 1,2 and 3 respectively, $S(t)$ the phase of service and $A(t)$ the phase of arrival at time t. With these the process $\{(N(t), N^*(t), S(t), A(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\Omega = \{\underline{0}, \underline{1}, \underline{2}, \ldots\}$, where

$$\underline{0} = \{(0, r); 1 \leq r \leq m\}$$

(in the level zero we need consider only the phase of arrival) and

$$\underline{i} = \{(i, j, k, r); i \geq 1, 1 \leq j \leq 3, 1 \leq k \leq n_j, 1 \leq r \leq m\}.$$

Thus the infinitesimal generator of this $CTMC$ is a $LIQBD$ and is of the form

$$Q = \begin{pmatrix} D_0 & A_{01} & & \\ A_{10} & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $A_{01} = \beta \otimes D_1$, $A_{10} = \ S^0 \otimes \ I_m$, $A_0 = \ I_n \otimes D_1$, $A_1 = S \oplus D_0$, $A_2 = \ S^0\beta \otimes \ I_m$.

2.1 Stability Condition

Consider $A(= A_0 + A_1 + A_2)$, the generator matrix of the Markov chain corresponding to the phase changes.

$$A = (S + \ S^0\beta) \oplus D = \begin{pmatrix} (p_1^0 \quad \boldsymbol{\beta}_1 + S_1) \oplus D & q_1^0 \quad \boldsymbol{\beta}_2 \otimes I_m & 0 \\ p_2^0 \quad \boldsymbol{\beta}_1 \otimes I_m & (q_2^0 \quad \boldsymbol{\beta}_2 + S_2) \oplus D & S_2^0\, \boldsymbol{\beta}_3 \otimes I_m \\ p_3^0 \quad \boldsymbol{\beta}_1 \otimes I_m & q_3^0 \quad \boldsymbol{\beta}_2 \otimes I_m & S_3 \oplus D \end{pmatrix} \tag{4}$$

Let $\pi = (\pi_1, \pi_2, \pi_3)$ be the steady-state probability vector of $(S + \ S^0\ \beta)$. Then

$$\pi\,(S + \ S^0\ \beta) = 0 and\ \pi\, e = 1. \tag{5}$$

From the relation $\pi\,(S + \ S^0\ \beta) = 0$ we have

$$\pi_1(p\,_1^0\,\boldsymbol{\beta}_1 + S_1) + \pi_2 p\,_2^0\,\boldsymbol{\beta}_1 + \pi_3 p\,_3^0\,\boldsymbol{\beta}_1 = 0, \tag{6}$$

$$\pi_1 q\,_1^0\,\boldsymbol{\beta}_2 + \pi_2(q\,_2^0\,\boldsymbol{\beta}_2 + S_2) + \pi_3 q\,_3^0\,\boldsymbol{\beta}_2 = 0, \tag{7}$$

$$\pi_2 \quad S_2^0\,\boldsymbol{\beta}_3 + \pi_3 S_3 = 0. \tag{8}$$

Multiplying equation (8) by e on right hand side we get

$$\pi_3\,_3^0 = \pi_2 \quad S_2^0\,. \tag{9}$$

Putting this in equation (6) yields

$$\pi_1\,_1^0 = -\frac{p}{q}\,\pi_2 S_2 e. \tag{10}$$

Substituting relations (9) and (10) in equation (7) gives

$$\pi_2(S_2^0\boldsymbol{\beta}_2 + S_2^0\boldsymbol{\beta}_2 + S_2) = 0. \quad (11)$$

This implies, for an arbitrary constant c

$$\pi_2 = c\boldsymbol{\beta}_2(-S_2)^{-1}. \quad (12)$$

Substituting for π_2 in relation (10) we have

$$\pi_1 = \frac{cp}{q}\boldsymbol{\beta}_1(-S_1)^{-1}. \quad (13)$$

Denote by $\delta = \boldsymbol{\beta}_2(-S_2)^{-1} S_2^0$ the probability that a customer starting with incorrect service leaves the system after getting correct service. Then equation (9) gives

$$\pi_3 = c\delta\boldsymbol{\beta}_3(-S_3)^{-1}. \quad (14)$$

From the normalizing condition $\pi e = 1$, the value of c is computed as

$$c = \left[\frac{p}{q}\mu'_1 + \mu'_2 + \delta\mu'_3\right]^{-1}. \quad (15)$$

Now from (1) and (5) we get the steady state probability vector of A as $\hat{\pi} = \pi \otimes \boldsymbol{\eta}$.

Theorem 2.1 *The system is stable if and only if*

$$\lambda < (\pi \otimes \boldsymbol{\eta})(S^0 \beta \otimes I_m)e. \quad (16)$$

Proof. The queueing system under study with the $LIQBD$ type generator given in (2) is stable if and only if rate of left drift is less than the rate of right drift (see Neuts [6]), that is,

$$\hat{\pi} A_0 e < \hat{\pi} A_2 e. \quad (17)$$

The left drift rate is $\hat{\pi}(I_n \otimes D_1)e$ which when simplified reduces to λ. Now, the right drift rate is $(\pi \otimes \boldsymbol{\eta})(S^0 \beta \otimes I_m)e$.

Let $\rho = \frac{\lambda}{(\pi \otimes \boldsymbol{\eta})(S^0 \beta \otimes I_m)e}$. Then from (16), we have $\rho < 1$.

2.2 Steady-State probability vector

A brief outline for the computation of the stationary probability vector of the system is as follows. Let $\mathbf{x}$ denote the steady-state probability vector of the generator Q. Then

$$\mathbf{x}Q = 0 and \mathbf{x}e = 1. \quad (18)$$

Assuming that the stability condition (16) holds and partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots)$ we obtain

$$\mathbf{x}_n = \mathbf{x}_1 R^{n-1}, n \geq 1 \quad (19)$$

where R is the minimal non negative solution to the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = 0$. The two boundary equations involving $\mathbf{x}_0$ are

$$\mathbf{x}_0 D_0 + \mathbf{x}_1 A_{10} = 0, \quad (20)$$

$$\mathbf{x}_0 A_{01} + \mathbf{x}_1[A_1 + RA_2] = 0 \quad (21)$$

These together with the normalizing condition in (18) gives

$$\mathbf{x}_1 = \mathbf{x}_0 V where V = -A_{01}[A_1 + RA_2]^{-1} \quad (22)$$

$$\mathbf{x}_0[I + V(I-R)^{-1}]e = 1. \quad (23)$$

To see how the system performs, it is instructive to define $\mathbf{y} = \sum_{i=1}^{\infty}\mathbf{x}_i$. Then $\mathbf{y} = (\mathbf{y}_1 \ \mathbf{y}_2 \ \mathbf{y}_3)$ where the $\mathbf{y}_i$, $i = 1,2,3$ indicates mode of service of the customer in service along with other system phases.

2.3 System Performance Measures

1. Probability that system is idle, $P_{idle} = \mathbf{x}_0 e = 1 - \rho$.
2. Rate of loss of customers, $R_{loss} = \mathrm{y}_2 \ S_2^0 = \lambda q(1-\delta)$.

3. Probability that a customer is lost, $P_{loss} = q(1-\delta)$.
4. Mean number of customers in the system, $\mu_{NS} = \sum_{i=1}^{\infty} i\mathbf{x}_i \mathrm{e}$.
5. Mean number of customers in the queue is given by $\mu_{NQ} = \sum_{i=2}^{\infty} (i-1)x_i \mathrm{e}$.
6. Probability that the server is serving in required mode, $P_C = \mathrm{y}_1\mathrm{e} + \mathrm{y}_3\mathrm{e} = \rho - \lambda q {\mu'}_2$.
7. Probability that the server is serving in unwanted mode, $P_I = \mathrm{y}_2\mathrm{e} = \lambda q {\mu'}_2$.
8. Rate at which customers leave with required service starting in desired service mode, $R_C = \mathrm{y}_1 \; S_1^0 = \lambda p$.
9. Rate at which customers leave with correct service starting with unwanted service, $R_I = \mathrm{y}_3 \; S_3^0 = \lambda q \delta$.
10. Expected waiting time in the system $W_S = \frac{\mu_{NS}}{\lambda}$.
11. We define the system reliability at any time as the probability of customers in service is in desired mode of service $p_{reliability} = (\mathrm{y}_1 + \mathrm{y}_3)\mathrm{e}$.

3 Case of Poisson arrival and phase type service

In this section we consider the system with Poisson arrival process and service times are phase type distributed (see Section 2). Then $\{(N(t), N^*(t), S(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{0, \underline{1}, \underline{2}, \dots\}$ where

$$\underline{i} = \{(i,j,k), 1 \leq j \leq 3, 1 \leq k \leq n_j\} for\, i \geq 1.$$

Thus the infinitesimal generator is of the form $Q' = \begin{pmatrix} -\lambda & \lambda\beta & & & \\ S^0 & S-\lambda I & \lambda I & & \\ & S^0\beta & S-\lambda I & \lambda I & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$.

Theorem 3.1 *The system is stable if and only if* $\rho' < 1$ *where*

$$\rho' = \lambda[p\mu_{1'} + q(\mu_{2'} + \delta\mu_{3'})]. \tag{24}$$

Proof. From the relation (17) we have $\lambda < \pi \;\; S^0 \;\; \beta\, e$ where $\pi = (\pi_1, \pi_2, \pi_3)$ (with π_i's as given in (12)-(14)) is the steady state probability vector of $S + \;\; S^0\beta$. The right drift $\pi \;\; S^0\beta\, e = \sum_{i=1}^{3} \pi_i \;\; S_i^0$.

Multiplying (6) by e on right hand side we get

$$\sum_{i=1}^{3} \pi_i \;\; S_i^0 = \frac{c}{q} - \frac{1}{p}\, \pi_i\, S_1 e = \frac{1}{p}\left(\frac{cp}{q}\right)\, \beta_1\, e \quad (from (13)) = \frac{c}{q}$$

where c is given in (15). Hence the condition for system stability is given by

$$\lambda < \frac{1}{p\mu_1' + q({\mu'}_2 + \delta\mu_3')} \tag{25}$$

The generator matrix corresponding to the phase changes is $S + \;\; S^0\,\beta$ and the stationary probability vector is $\pi = (\pi_1, \pi_2, \pi_3)$.

Theorem 3.2 *The steady-state probability vector* $x = (x_0, x_1, x_2, \cdots)$ *of* Q' *is given by*

$$x_0 = 1 - \rho', \mathrm{x}_i = (1-\rho')\,\beta\, R^i, i \geq 1, \tag{26}$$

where R is given by

$$R = \lambda \begin{bmatrix} \lambda I - \lambda p\mathrm{e}\,\beta_1 - S_1 & -\lambda q\mathrm{e}\,\beta_2 & 0 \\ -\lambda p\mathrm{e}\,\beta_1 & \lambda I - \lambda q\mathrm{e}\,\beta_2 - S_2 & - \;\; S_2^0\;\beta_3 \\ -\lambda p\mathrm{e}\,\beta_1 & -\lambda q\mathrm{e}\,\beta_2 & \lambda I - S_3 \end{bmatrix}^{-1}. \tag{27}$$

Proof. Let x be the steady-state probability vector of Q'. Then $xQ' = 0$ and $xe = 1$.

The steady-state equations are given by

$$-\lambda x_0 + x_1 S^0 = 0, \tag{28}$$

$$\lambda x_0 \boldsymbol{\beta} + x_1(S - \lambda I) + x_2 S^0 \boldsymbol{\beta} = 0, \tag{29}$$

$$\lambda x_{i-1} + x_i(S - \lambda I) + x_{i+1} S^0 \boldsymbol{\beta} = 0, for i \geq 2. \tag{30}$$

From (28) we have

$$x_1 S^0 = \lambda x_0. \tag{31}$$

Multiplying equations (29) and (30) by the column vector e on the right hand side leads to

$$x_{i+1} S^0 = \lambda x_i e for i \geq 1. \tag{32}$$

Since $x_{i+1} S^0 \boldsymbol{\beta} = \lambda x_i \mathcal{B} for i \geq 1$ where $\mathcal{B} = e.\boldsymbol{\beta}$, from (29) and (30) we obtain

$$x_1(\lambda I - \lambda \mathcal{B} - S) = \lambda x_0 \boldsymbol{\beta} \tag{33}$$

and

$$x_i(\lambda I - \lambda \mathcal{B} - S) = \lambda x_{i-1}, for i \geq 2. \tag{34}$$

Denoting $(\lambda I - \lambda \mathcal{B} - S)$ by $\mathcal{K}$, relation (33) takes the form $x_1 = \lambda x_0 \boldsymbol{\beta} \mathcal{K}^{-1}$, provided $\mathcal{K}$ is invertible. We now prove the nonsingularity of $\mathcal{K}$.

Let the vector u be in the left kernal of $\mathcal{K}$. Then

$$\lambda u - uS - \lambda(ue)\boldsymbol{\beta} = 0. \tag{35}$$

Suppose $ue = 0$. Then (35) reduces to $u(\lambda I - S) = 0$. But $(\lambda I - S)$ is nonsingular and hence $u = 0$.

If $ue \neq 0$, normalize u by setting $ue = 1$. Post multiplying (35) by e gives

$$uS^0 = 0. \tag{36}$$

Substituting for ue, (35) reduces to $u = \lambda \boldsymbol{\beta}(\lambda I - S)^{-1}$.

From (36) we have

$$\lambda \boldsymbol{\beta}(\lambda I - S)^{-1} S^0 = 0. \tag{37}$$

In (37) $\boldsymbol{\beta}(\lambda I - S)^{-1} S^0$ is the Laplace-Stieltjes transform at $s = \lambda (> 0)$, of the probability distribution $F(t) = 1 - \boldsymbol{\beta} \exp(St) e$ for $t \geq 0$. Therefore (37) cannot hold and hence $u = 0$. Thus $\mathcal{K}$ is nonsingular.

The irreducibility of the representation $(\boldsymbol{\beta}, S)$ leads to the irreducibility of $\mathcal{K}$, so that the matrix R in (27) is positive.

We have $sp(R) < 1$, if $\rho' < 1$. Therefore the quantity x_0 is given by the normalizing equation

$$x_0 + x_0 \boldsymbol{\beta} R (I - R)^{-1} e = 1.$$

Substitution for R leads to

$$x_0 - \lambda x_0 \boldsymbol{\beta}(\lambda \mathcal{B} + S)^{-1} e = 1. \tag{38}$$

The inverse of $(\lambda \mathcal{B} + S)$ is calculated as

$$\begin{aligned} (\lambda \mathcal{B} + S)^{-1} &= S^{-1}(I + \lambda \mathcal{B} S^{-1})^{-1} = S^{-1} \textstyle\sum_{n=0}^{\infty} (-1)^n \lambda^n (\mathcal{B} S^{-1})^n \\ &= S^{-1}[I - \lambda[\textstyle\sum_{n=0}^{\infty} (-1)^n \lambda^n (\mathcal{B} S^{-1})^n] \mathcal{B} S^{-1}] = S^{-1}[I - \lambda \textstyle\sum_{n=0}^{\infty} \rho'^n \mathcal{B} S^{-1}] \\ &= S^{-1}[I - \lambda(1 - \rho')^{-1} \mathcal{B} S^{-1}]. \end{aligned}$$

From (38) we have

$$\begin{aligned} x_0 - \lambda x_0 \boldsymbol{\beta}(\lambda \mathcal{B} + S)^{-1} e &= x_0 - \lambda x_0 \boldsymbol{\beta}[S^{-1}(I - \lambda(1 - \rho')^{-1} \mathcal{B} S^{-1})] e \\ &= x_0 - \lambda x_0 \boldsymbol{\beta} S^{-1} e + \lambda^2 x_0 (1 - \rho)^{-1} \boldsymbol{\beta} S^{-1} \mathcal{B} S^{-1} e \\ &= x_0 + \rho' x_0 + \rho'^2 (1 - \rho') x_0 = (1 - \rho') x_0 = 1, \end{aligned}$$

so that $x_0 = (1 - \rho')$.

Letting $y = \sum_{i=1}^{\infty} x_i$, it is obtained that $y = \rho' \pi$. In the sequel partition $y = (y_1, y_2, y_3)$, so that $y_i = \rho' \pi_i, 1 \leq i \leq 3$.

4 Poisson arrival with exponential service

In this section we consider customers arrive according to the Poisson process with rate λ and desired (correct) service time follows exponential distribution but the undesired (incorrect) service follows phase type distribution. Let $N(t)$ be the number of customers in the system $N^*(t)$ the type of service and $S(t)$ the phase of service at time t. Then $\{(N(t), N^*(t), S(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{0, \underline{1}, \underline{2}, \dots\}$ where

$$\underline{i} = \{(i,1,0),(i,3,r+1)\} \cup \{(i,2,j), 1 \le j \le r\} for i \ge 1.$$

Thus the infinitesimal generator is of the form

$$Q = \begin{array}{c|cccccc} & 0 & \underline{1} & \underline{2} & \underline{3} & \cdots & \cdots \\ \hline 0 & -\lambda & b_0 & & & & \\ \underline{1} & c_0 & A_1 & A_0 & & & \\ \underline{2} & & A_2 & A_1 & A_0 & & \\ \vdots & & & \ddots & \ddots & \ddots & \end{array}$$

$$where\ b_0 = \lambda(p, q\boldsymbol{\beta}, 0), \quad c_0 = \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix}, A_0 = \lambda I$$

$$A_1 = \begin{pmatrix} -\lambda-\mu & 0 & 0 \\ 0 & \tilde{S} - \lambda I & \tilde{S}_2^0 \\ 0 & 0 & -\lambda-\mu \end{pmatrix}, A_2 = \begin{pmatrix} \mu p & \mu q\boldsymbol{\beta} & 0 \\ p\ \tilde{S}_1^0 & q\ \tilde{S}_1^0\boldsymbol{\beta} & 0 \\ \mu p & \mu q\boldsymbol{\beta} & 0 \end{pmatrix} with\ \tilde{S}e + \tilde{S}_1^0 + \tilde{S}_2^0 = 0.$$

4.1 Stability condition

Consider $A = A_0 + A_1 + A_2$

$$= \begin{pmatrix} -\mu q & \mu q\boldsymbol{\beta} & 0 \\ p\ \tilde{S}_1^0 & \tilde{S} + q\ \tilde{S}_1^0\boldsymbol{\beta} & \tilde{S}_2^0 \\ \mu p & \mu q\boldsymbol{\beta} & -\mu \end{pmatrix} \tag{39}$$

the generator matrix of the Markov chain corresponding to the phase changes. Let

$\Pi = (\pi_0, \widehat{\boldsymbol{\pi}}, \pi_{r+1})$ be the steady state probability matrix of A. Solving the relations

$$\Pi A = 0, \quad \Pi e = 1 \tag{40}$$

we obtain

$$-\mu q\ \pi_0 + p\widehat{\boldsymbol{\pi}}\ \ \tilde{S}_1^0 + \mu p\ \pi_{r+1} = 0 \tag{41}$$

$$\mu q \pi_0 \boldsymbol{\beta} + \widehat{\boldsymbol{\pi}}(\ \tilde{S} + q\ \tilde{S}_1^0\ \boldsymbol{\beta}) + \mu q\ \pi_{r+1}\boldsymbol{\beta} = 0 \tag{42}$$

$$\widehat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0 - \mu\pi_{r+1} = 0 \tag{43}$$

Equation (43) gives

$$\mu\pi_{r+1} = \widehat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0 \tag{44}$$

Putting this in equation(41),

$$\mu q\ \pi_0 = p\big(\widehat{\boldsymbol{\pi}}\ \ \tilde{S}_1^0 + \widehat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0\big) \tag{45}$$

Substituting these in equation(42) and simplifying we get

$$\widehat{\boldsymbol{\pi}}(S + q\ \tilde{S}_1^0\ \boldsymbol{\beta}) + p\widehat{\boldsymbol{\pi}}\ \big(\ \tilde{S}_1^0\beta + \ \tilde{S}_2^0\ \boldsymbol{\beta}\big) + q\widehat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0\ \boldsymbol{\beta} = 0$$

$$\Rightarrow \widehat{\boldsymbol{\pi}}\ \big(\ \tilde{S} + \ \tilde{S}_1^0\ \boldsymbol{\beta} + \ \tilde{S}_2^0\ \boldsymbol{\beta}\big) = 0$$

so that

$$\widehat{\boldsymbol{\pi}} = c\ \boldsymbol{\beta}(-\ \tilde{S}\ )^{-1} \tag{46}$$

c being a constant and is computed from the normalizing condition. Let δ be the probability that a customer getting correct service from incorrect services and η the probability of staying back in incorrect services. Then

$$\delta = \boldsymbol{\beta}\ (-\ \tilde{S}\ )^{-1}\ \ \tilde{S}_2^0 \tag{47}$$

and

$$\eta = \left(\boldsymbol{\beta}(-\ \tilde{S}\ )^{-1}e\right)^{-1} \tag{48}$$

Then the probability that a customer leaves the system without getting required service is

$$1 - \delta = \boldsymbol{\beta}\ (-\ \tilde{S}\ )^{-1}\ \ \tilde{S}_1^0 \tag{49}$$

and the mean time a customer stay back in incorrect services is

$$\frac{1}{\eta} = \left(\boldsymbol{\beta}(-\ \tilde{S}\ )^{-1}e\right) \tag{50}$$

The normalizing equation is $\pi_0 + \widehat{\boldsymbol{\pi}}\ e + \pi_{r+1} = 1$. Substituting for the components of Π which are now computed as

$$\pi_0 = \frac{pc}{\mu q},\ \hat{\boldsymbol{\pi}}\ e = \frac{c}{\eta},\ \pi_{r+1} = \frac{c\delta}{\mu} \tag{51}$$

we get $\frac{pc}{\mu q} + \frac{c}{\eta} + \frac{c\delta}{\mu} = 1$ which shows

$$c = \frac{\mu q \eta}{p\eta + \mu q + \delta q \eta}. \tag{52}$$

Theorem 4.1 *The system is stable if and only if* $\lambda < \frac{1}{q}\ c$.

Proof. The condition for the stability of the system is $\Pi A_0 e < \Pi A_2 e$. Simplification gives $\Pi A_0 e = \lambda$. Now $A_2 e = \begin{pmatrix} \mu & S_1^0 & \mu \end{pmatrix}^T$. Therefore $\Pi A_2 e = \mu \pi_0 + \hat{\boldsymbol{\pi}}(\ \tilde{S}\,_1^0 + \ \tilde{S}\,_2^0)$ Substituting for $\mu\pi_0$, right hand side becomes $\frac{1}{q}\hat{\boldsymbol{\pi}}(\ \tilde{S}\,_1^0 + \ \tilde{S}\,_2^0)$. Using equation(46) and the fact that $(\ \tilde{S}\ )^{-1}(\ \tilde{S}\,_1^0 + \tilde{S}\,_2^0) = e$, the result follows. Hence the system is stable if $\rho < 1$ where $\rho = \lambda\ \frac{q}{c}$.

4.2 Steady-State probability Vector

Let the steady state probability vector x of Q be $x = (x^*, \mathbf{x}(1), \mathbf{x}(2), \ldots)$ be such that $xQ = 0, xe = 1$. Partitioning gives $\mathbf{x}(i) = (x_0(i), \frown\ x(i), x_{r+1}(i))$. The relation $xQ = 0$ gives the following system of equations.

$$-\lambda x^* + \mathbf{x}(1)c_0 = 0 \tag{53}$$

$$x^* b_0 + \mathbf{x}(1)A_1 + \mathbf{x}(2)A_2 = 0 \tag{54}$$

$$\text{For} i \geq 1, \mathbf{x}(i-1)A_0 + \mathbf{x}(i)A_1 + \mathbf{x}(i+1)A_2 = 0 \tag{55}$$

From the matrix geometric structure we obtain

$$\mathbf{x}(i) = \mathbf{x}(1)R^{i-1}, i \geq 1 \tag{56}$$

where R is the minimal non negative solution to the matrix quadratic equation $R^2 A_2 + RA_1 + A_0 = O$. Equation (53) shows

$$x^* = \frac{1}{\lambda}\mathbf{x}(1)c_0. \tag{57}$$

Equation (54) together with normalizing condition gives

$$x^* b_0 + \mathbf{x}(1)(A_1 + RA_2) = 0 \tag{58}$$

$$\text{subjectto} x^* e + \mathbf{x}(1)(I - R)^{-1} e = 1. \tag{59}$$

Substituting for x^*,

$$\mathbf{x}(1)\left(A_1 + RA_2 + \frac{1}{\lambda}c_0 b_0\right) = 0 \tag{60}$$

$$\text{subjectto} \mathbf{x}(1)\left(\frac{1}{\lambda}c_0 + (I - R)^{-1} e\right) = 1. \tag{61}$$

But $c_0 b_0 = \lambda A_2$ which implies

$$\mathbf{x}(1)(A_1 + RA_2 + A_2) = 0 \tag{62}$$

$$\text{subjectto} \mathbf{x}(1)\left(\frac{1}{\lambda}c_0 + (I - R)^{-1} e\right) = 1. \tag{63}$$

4.2.1 *Computation of* $\boldsymbol{R}$

R can computed explicitly along the following lines.

We have

$$A_2 = \begin{pmatrix} \mu p & \mu q \boldsymbol{\beta} & 0 \\ p\, \tilde{S}_1^0 & q\, \tilde{S}_1^0 \boldsymbol{\beta} & 0 \\ \mu p & \mu q \boldsymbol{\beta} & 0 \end{pmatrix} = \begin{bmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{bmatrix} \begin{bmatrix} p & q\boldsymbol{\beta} & 0 \end{bmatrix} \tag{64}$$

so that

$$A_2 e = \begin{bmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{bmatrix} = c_0 \tag{65}$$

Also from the relation $RA_2 e = A_0 e$, we obtain

$$RA_2 e = \lambda e. \tag{66}$$

Now, $R^2 A_2 = R^2 \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} (p \quad q\boldsymbol{\beta} \quad 0) = R^2 A_2 e (p \quad q\boldsymbol{\beta} \quad 0)$.

Substituting for RA_2 from (66), we get

$$R^2 A_2 = R\lambda e (p \quad q\boldsymbol{\beta} \quad 0) \tag{67}$$

Therefore

$$\lambda Re \quad (p \quad q\boldsymbol{\beta} \quad 0) + RA_1 + \lambda I = 0. \tag{68}$$

This gives

$$R = \lambda \begin{pmatrix} \mu + \lambda q & -\lambda q \boldsymbol{\beta} & 0 \\ -\lambda p e & \lambda I - \lambda q e \boldsymbol{\beta} - \quad \tilde{S} & - \quad \tilde{S}_2^0 \\ -\lambda p & -\lambda q \boldsymbol{\beta} & \lambda + \mu \end{pmatrix}^{-1}. \tag{69}$$

Lemma 4.2 $x^* = 1 - \rho$ *so that* $\boldsymbol{x}(1)(I - R)^{-1} e = \rho$.

Proof. Multiplying by e on the right side of equation (54) and simplifying we get the relation

$$\lambda x^* + \mathbf{x}(1) \begin{pmatrix} -\lambda - \mu \\ \tilde{S} - \lambda I + \quad \tilde{S}_2^0 \\ -\lambda - \mu \end{pmatrix} + \mathbf{x}(2) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = 0. \tag{70}$$

Equation (53) gives

$$\lambda x^* = \mathbf{x}(1) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix}. \tag{71}$$

Putting this in (70) the following relation is obtained.

$$\mathbf{x}(2) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = \lambda \mathbf{x}(1) e. \tag{72}$$

Multiplying equation(55) on right side by $\mathbf{e}$ and recursive use of the relation results in

$$\mathbf{x}(i) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = \lambda \mathbf{x}(i-1) e \quad \text{for } i \geq 3. \tag{73}$$

Adding (71), (72) and (73)

$$\sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = \lambda. \tag{74}$$

Adding the system of equations (55) with equation (54) and using the fact that $x^* b_0 = \mathbf{x}(1) A_2$ we get

$$\sum_{i=1}^{\infty} \mathbf{x}(i) \ A = 0. \tag{75}$$

But the relation (40) says

$$\sum_{i=1}^{\infty} \mathbf{x}(i) \ = d \ \Pi \quad \text{forsomeconstant } c \tag{76}$$

which in turn gives

$$\sum_{i=1}^{\infty} \mathbf{x}(i) \ = (1 - x^*) \ \Pi \tag{77}$$

Multiplying on the right side by $\begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix}$ and using the relation in (??)

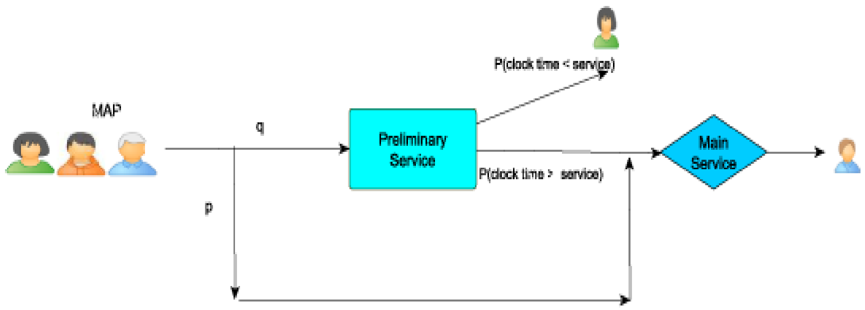
$$\sum_{i=1}^{\infty} \mathbf{x}(i) \ \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = (1 - x^*) \ \frac{\lambda}{\rho} \tag{78}$$

The result follows from (74) and (78).

4.3 System Performance measures

1. Probability that the system is idle $= x^*$
2. Rate of loss $= \sum_{i=1}^{\infty} \frown x(i) \ \tilde{S}_1^0 = \lambda q(1-\delta)$
3. Probability of loss $= q(1-\delta)$
4. Mean number of customers in the system $= \sum_{i=1}^{\infty} i\mathbf{x}(i)e = \mathbf{x}(1)(I-R)^{-2}e$
5. Mean number of customers in the queue $= \sum_{i=1}^{\infty} (i-1)\mathbf{x}(i)e = \mathbf{x}(1)(I-R)^{-2}e - \mathbf{x}(1)(I-R)^{-1}e$
6. Probability that the server is busy serving in correct mode
$$\sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{x}(1)(I-R)^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \rho(\pi_0 + \pi_{r+1}) = \rho - \frac{\lambda q}{\eta}$$
7. Probability that the server is busy serving in correct mode
$$\sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} 0 \\ e \\ 0 \end{pmatrix} = \rho \frown \pi e = \frac{\lambda q}{\eta}$$

5 An illustration



In this section we consider a queueing model consisting of two service stations - preliminary service and main service. Customers arrive to this system according to a MAP (Markovian Arrival Process) with representation (D_0, D_1) of order m. A customer, which taken for service is directly selected for main service with probability p or to the preliminary service with probability $q(= 1 - p)$. A threshold clock starts ticking if a customer enters to preliminary service. When the duration of preliminary service exceeds the threshold clock, the customer move out of the system, else he goes to main service. The threshold clock follows exponential distribution with parameter ζ. Service times of the customers at these stations follow phase type distributions with representation $(\boldsymbol{\alpha}, S_P)$, $(\boldsymbol{\gamma}, S_M)$ and of order a, b respectively. Write $S_P^0 + \zeta e = -S_P e$ and $S_M^0 = -S_M e$ where e is a column vector of 1's of appropriate order. Hence service time of a customer can be modeled as a phase type distribution with representation $(\boldsymbol{\xi}, U)$ of order $a + 2b$ such that $Ue + U^0 = 0$ where
$$\boldsymbol{\xi} = (p\boldsymbol{\gamma} \quad q\boldsymbol{\alpha} \quad 0)$$

$$U = \begin{pmatrix} S_M & 0 & 0 \\ 0 & S_P & S_P^0\boldsymbol{\gamma} \\ 0 & 0 & S_M \end{pmatrix}, U^0 = \begin{pmatrix} S_M^0 \\ \zeta e \\ S_M^0 \end{pmatrix}.$$

Let $N(t), N^*(t), S(t), A(t)$ denote respectively the number of customers in the system, nature of service, phase of service and phase of arrival at time t.

$$N^*(t) = \begin{cases} 1 & mainservice \\ 2 & preliminaryservice \\ 3 & onethatcomefrompreliminaryservice \end{cases}$$

The process $\Omega = \{(N(T), N^*(t), S(t), A(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{(n, i, j, k); i = 1,3, 1 \leq j \leq b, 1 \leq k \leq m\} \cup \{(n, 2, j, k); 1 \leq j \leq a, 1 \leq k \leq m\}$ for $n \geq 1$. Note that when $N(t) = 0$, the only other component in the state vector is $A(t)$. Thus the infinitesimal generator of Ω is of the form

$$Q^* = \begin{pmatrix} D_0 & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{79}$$

where $A_{01} = \boldsymbol{\xi} \otimes D_1, A_{10} = U^0 \otimes I_m, A_0 = I_{a+2b} \otimes D_1, A_1 = U \oplus D_0, A_2 = U^0 \boldsymbol{\xi} \otimes I_m$.

The infinitesimal generator Q^* given by (79) is of the same form as Q of the model described initially. Thus the analysis of the Markov chain with infinitesimal generator Q^* can be done in the same way as for Q.

The significance of this model is as follows: customer arriving to a single server belong to two categories, though they join the same. Only while taken for service the category will be revealed. Call them category 1 and category 2, respectively. Category 1 are qualified for the main service without undergoing preliminary service. However, category 2 have to be given the preliminary service before admitted to mean service. However, if such customers do not get service in preliminary before realization of the timer (random clock), they get disqualified and so leave the system forever. On the other those among category 2, completing service successfully in preliminary are immediately admitted to main service. On completion of that service such customers leave the system.

Remark 5.1 *In telecommunication it is this type of situation that is often encountered. Packages have to identify the server in idle state; then wait for a while. But in the mean time another message may get through, making the server busy. Then the customer (packet) under consideration has to go through a series of contention windows. These passages could be regarded as unwanted service. In case the process of going through contention windows exceeds a threshold time limit, the message will not get served.*

Remark 5.2 *The problem discussed in Madan [3] and Medhi [5] could be arrived at from our model as follows. Suppose that we reverse the order of preliminary and main service, that is, main service first and preliminary (hereafter we call it optional) service next. Then after completion of main service, the customer asks for an optional service with probability* $1 - q$ *(this optional service time has exponential distribution in Madan [3]). With probability q, the customer leaves the system immediately after main service completion. This model is also the same as a queue with instantaneous feedback after a service and immediate commencement of his service (feedback restricted to one). This feedback policy is referred to as queues with instantaneous feedback as head of the queue.*

6 Numerical illustration

The following numerical illustration is based on the description in Section 2.

We fix parameters $n_1 = 2, n_2 = 3, n_3 = 4, \boldsymbol{\beta}_1 = (0.4 \quad 0.6), \boldsymbol{\beta}_2 = (0.3 \quad 0.5 \quad 0.2), \boldsymbol{\beta}_3 = (0.2 \quad 0.3 \quad 0.3 \quad 0.2)$,

$$S_1 = \begin{bmatrix} * & 6 \\ 8 & * \end{bmatrix}, S_1^0 = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \; with \; S_1 e + S_1^0 = 0,$$

$$S_2 = \begin{bmatrix} * & 5 & 5 \\ 6 & * & 6 \\ 5 & 7 & * \end{bmatrix}, S_2^0 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, S_2^0 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} with\ S\_2\ e + S\_2\text{^}0 + S\_2\text{^}0 = 0,$$

$$S_3 = \begin{bmatrix} * & 7 & 8 & 9 \\ 6 & * & 7 & 7 \\ 6 & 6 & * & 6 \\ 8 & 7 & 6 & * \end{bmatrix}, S_3^0 = \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} \ with\ S_3 e + S_3^0 = 0.$$

For the arrival process, we consider the following two sets of values for D_0 and D_1 as follows. The arrival processes labeled $MNCA$ and $MPCA$ respectively, have negative and positive correlation for two successive inter-arrival times with values -0.48891 and 0.48891. The standard deviation of the inter-arrival times of these two arrival processes are, respectively, 0.2819 and 0.2819.

1. MAP with negative correlation ($MNCA$):

$$D_0 = \begin{pmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.05011 & 0 & 4.96099 \\ 1117.4625 & 0 & 11.2875 \end{pmatrix}$$

p	P_{loss}	μ_{NS}	P_C	P_I	R_C	R_I	W_S
.4	0.2136	7.5229	0.5242	0.3921	2	1.9320	1.5046
.5	0.1780	4.9744	0.5483	0.3267	2.5	1.6100	0.9949
.6	0.1424	3.6690	0.5724	0.2614	3	1.2880	0.7338
.7	0.1068	2.8654	0.5965	0.1960	3.5	0.9660	0.5731
.8	0.0712	2.3138	0.6206	0.1307	4	0.6440	0.4628
.9	0.0356	1.9069	0.6447	0.0653	4.5	0.3220	0.3814

Table 1: Effect of p for $MNCA$

2. MAP with positive correlation ($MPCA$):

$$D_0 = \begin{pmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 4.96099 & 0 & 0.05011 \\ 11.2875 & 0 & 1117.4625 \end{pmatrix}$$

p	P_{loss}	μ_{NS}	P_C	P_I	R_C	R_I	W_S
.4	0.2136	546.8179	0.5242	0.3921	2	1.9320	109.3646
.5	0.1780	349.9587	0.5483	0.3267	2.5	1.6100	69.9924
.6	0.1424	250.7699	0.5724	0.2614	3	1.2880	50.1545
.7	0.1068	191.0008	0.5965	0.1960	3.5	0.9660	38.2005
.8	0.0712	151.0402	0.6206	0.1307	4	0.6440	30.2083
.9	0.0356	122.4351	0.6446	0.0653	4.5	0.3220	24.4873

Table 2: Effect of p for $MPCA$

The output in Tables 1 and 2 are on expected lines. Note that P_{loss} decreases with increasing value of p. The value of $P_C(R_C)$ steadily increases with p and values of $P_I(R_I)$ and W_S decrease with increase in value of p, as expected.

The main comparison in Tables 1 and 2 is between values of μ_{NS} in $MNCA$ and $MPCA$. Both decrease with increase in value of p. However, $MNCA$ has much smaller values compared to their $MPCA$ counter parts. This indicates that positive correlation in the arrival process results in accumulation of large number of customers in the system.

7 $M/G/1$ Model

In this section we consider an $M/G/1$ system with two service stations – preliminary service and main service. Customers arrive to this system according to a Poisson process with rate λ. A customer, when taken for service, is directly selected for main service with probability p or to the preliminary service with probability q $(= 1 - p)$. A threshold clock starts ticking if a customer enters to preliminary service. When the duration of preliminary service exceeds the threshold clock, the customer moves out of the system, else he goes to main service. The threshold clock follows exponential distribution with parameter ζ. Here the service times, V_p, V_m of the preliminary and main services are independent having general distributions with distribution function $G_1(.), G_2(.)$, LST $G_1^*(.), G_2^*(.)$ respectively.

The (total) service time V of a unit is

$$V = \begin{cases} V_f & withprobabilityq \bullet P(G_1(.) > \exp(\zeta)) \\ V_p & withprobabilityq \bullet P(G_1(.) < \exp(\zeta)) \\ V_m & withprobabilityp \end{cases}$$

where V_f is the duration of threshold clock realization.

Thus

$$G(t) = P(V \leq t) = q\left[\int_0^t \zeta e^{-\zeta u}(1 - G_1(u))du + \int_0^t e^{-\zeta u} G_1(u) dG_2(t-u)\right] + p\int_0^t dG_2(u)$$

and LST $G^*(s)$ of V is given by $G^*(s) = \int_0^\infty e^{-st} dG(t)$.

Remark 7.1 *This modelling closely resembles the protocol IEEE 802.11. This is so because of a message generated has to wait before checking for idle server; if server is busy it has to go through a series of contention windows and then look for idle server. In case this process takes longer duration than the life of message (before its significance is lost), then the message does not serve any purpose. In the opposite case it is transmitted before its expiry time.*

Remark 7.2 *Assume the random clock to be of infinite duration (ie., its rate of realization goes to zero). Now interchange the roles of preliminary and main services (in this case, we call the preliminary service, which is the second one now, as optional service). Invariably main service is given for all customers. Thus the main service is followed by an optional service to which customers, on completion of main service, proceed with probability q. Then our model reduces to Madan [3] with exponentially distributed optional service and to Medhi [5] in the case of arbitrarily distributed optional service time.*

Transient solution

The supplementary variable technique (see Cox [1], Medhi [4]) could be used to get the transient solution. Denote by $h(x) = \frac{dG(x)}{1-G(x)}$, the hazard function of the service time distribution $G(.)$ and the probability density function of V is given by

$$g(x) = h(x)\exp\{-N(x)\}$$

where

$$N(x) = \int_0^x h(u)du \qquad \left(N(0) = 0 and \frac{d}{dx}N(x) = h(x)\right).$$

If V is the total service time, then $h(x)dx = P$(service will be completed in $(x, x+dx)$ given that service time exceeds x) and $E(V) = \int xg(x)dx = -G^{*(1)}(0)$.

The supplementary variable $X(t)$ considered is defined below. Let

$$\begin{aligned} N(t) &= systemsizeattimet \\ X(t) &= timealreadyspentinserviceuptotof aunitreceivingservice \\ p_n(t) &= P(N(t) = n) with p_0(0) = 1 \\ p_n(t,x)dx &= P(N(t) = n, x \leq X(t) < x + dx),\ n \geq 1 \end{aligned}$$

$$p_n(t) = \int_0^\infty p_n(t,x)dx, \quad Q(t,z) = \sum_{n=0}^\infty p_n(t)z^n, \ Q(t,x,z) = \sum_{n=1}^\infty p_n(t,x)z^n$$

Now we have

$$p_0(t+\delta t) = [1-\lambda\delta t + o(\delta t)]p_0(t) + \int_0^\infty p_1(t,x)h(x)dx\delta t.$$

$$As\delta t \to 0, \quad \frac{\partial}{\partial t}p_0(t) = -\lambda p_0(t) + \int_0^\infty p_1(t,x)h(x)dx. \tag{80}$$

$$For\delta x > 0, \; p_1(t+\delta t, x+\delta x) = [1-\lambda\delta t + o(\delta t)][1-h(x)\delta x + o(\delta x)]p_1(t,x).$$

Subtracting and adding a term $p_1(t, x+\delta x)$ to the LHS, then dividing by $\delta t(\delta x)$ and taking as $\delta t \to 0(\delta x \to 0)$, we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)p_1(t,x) = -(\lambda + h(x))p_1(t,x). \tag{81}$$

$$Forn \geq 0, \; \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)p_n(t,x) = -(\lambda + h(x))p_n(t,x) + \lambda p_{n-1}(t,x). \tag{82}$$

We have the following boundary conditions:

$$p_1(t,0) = \int_0^\infty p_2(t,x)h(x)dx + \lambda p_0(t) \tag{83}$$

and

$$p_n(t,0) = \int_0^\infty p_{n+1}(t,x)h(x)dx, \qquad n \geq 2. \tag{84}$$

Multiplying (82) by $z^n, n = 2,3,\ldots$ and (81) by z, then adding all the terms we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\sum_{n=1}^\infty p_n(t,x)z^n = -(\lambda + h(x))\sum_{n=1}^\infty p_n(t,x) + \lambda\sum_{n=2}^\infty p_{n-1}(t,x) \tag{85}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)Q(t,x,z) = -(\lambda - \lambda z + h(x))Q(t,x,z). \tag{86}$$

Now multiplying (84) by $z^n, n = 2,3,\ldots$ and (83) by z, then adding the terms we have

$$Q(t,0,z) = \int_0^\infty \left(\sum_{n=1}^\infty p_{n+1}(t,x)z^n\right)h(x)dx + \lambda z p_0(t). \tag{87}$$

Now

$$\begin{aligned}
\int_0^\infty \left(\sum_{n=1}^\infty p_{n+1}(t,x)z^n\right)h(x)dx &= \int_0^\infty \left(\frac{1}{z}\right)\sum_{n=1}^\infty p_{n+1}(t,x)z^{n+1}h(x)dx \\
&= \int_0^\infty \left(\frac{1}{z}\right)\left[\sum_{n=1}^\infty p_n(t,x)z^n - p_1(t,x)z\right]h(x)dx \\
&= \left(\frac{1}{z}\right)\int_0^\infty [Q(t,x,z) - p_1(t,x)z]h(x)dx \\
&= \left(\frac{1}{z}\right)\left[\int_0^\infty Q(t,x,z)h(x)dx - z(p'_0(t) + \lambda p_0(t))\right] by (80)
\end{aligned}$$

Thus (87) reduces to

$$\begin{aligned}
Q(t,0,z) &= \left(\frac{1}{z}\right)\left[\int_0^\infty Q(t,x,z)h(x)dx - z(p'_0(t) + \lambda p_0(t))\right] + \lambda z p_0(t) \\
&= \left(\frac{1}{z}\right)\left[\int_0^\infty Q(t,x,z)h(x)dx - z(p'_0(t) + \lambda p_0(t)) + \lambda z^2 p_0(t)\right]
\end{aligned}$$

$$zQ(t,0,z) = \int_0^\infty Q(t,x,z)h(x)dx - zp'_0(t) + \lambda z(z-1)p_0(t). \tag{88}$$

The partial differential equation (86) can be solved using the boundary condition (88) and the normalizing condition $\sum_{n=0}^\infty p_n(t) = 1$.

7.1 Steady state distribution

Let

$$lim_{t\to\infty} p_n(t) = p_n, \; n \geq 0$$

and

$$\begin{aligned}
\lim_{t\to\infty} p_n(t,x) &= p_n(x), \qquad x > 0, n \geq 1 \\
&= p_0(x) = 0, \quad x > 0.
\end{aligned}$$

Then $\{p_n, n \geq 0\}$ gives the distribution of the general time system size.

Let

$$\begin{aligned} Q(x,z) &= \sum_{n=1}^{\infty} p_n(x) z^n = \sum_{n=1}^{\infty} \left[\lim_{t\to\infty} p_n(t,x)\right] z^n \\ &= \lim_{t\to\infty} \left[\sum_{n=1}^{\infty} p_n(t,x) z^n\right] = \lim_{t\to\infty} Q(t,x,z) \end{aligned}$$

and

$$Q(z) = \int_0^\infty Q(x,z)dx.$$

Then

$$(80) \Rightarrow \lambda p_0 = \int_0^\infty p_1(x)h(x)dx \tag{89}$$

$$(81) and (8182) \Rightarrow \frac{\partial}{\partial x} p_n(x) = -(\lambda + h(x))p_n(x) + \lambda p_{n-1}(x),\ n \geq 1 \tag{90}$$

$$(83) \Rightarrow p_1(0) = \int_0^\infty p_2(x)h(x)dx + \lambda p_0 \tag{91}$$

$$(84) \Rightarrow p_n(0) = \int_0^\infty p_{n+1}(x)h(x)dx,\ n \geq 2. \tag{92}$$

The partial differential equation (86) and the boundary condition (88) reduces to

$$\frac{d}{dx} Q(x,z) = -(\lambda - \lambda z + h(x))Q(x,z) \tag{93}$$

$$zQ(0,z) = \int_0^\infty Q(x,z)h(x)dx + \lambda z(z-1)p_0 \tag{94}$$

and

$$p_0 + Q(1) = 1. \tag{95}$$

From relation (93)

$$\begin{aligned} \int \frac{dQ(x,z)}{Q(x,z)} &= \int -\big(\lambda - \lambda z + h(x)\big)dx \\ \log\big(Q(x,z)\big) &= logc\big(-\lambda(1-z)x - N(x)\big) \\ Q(x,z) &= c\exp\big(-\lambda(1-z)x - N(x)\big) \\ Q(0,z) &= c \end{aligned}$$

$$Q(x,z) = Q(0,z)\exp(-\lambda(1-z)x - N(x)) \tag{96}$$

Substituting (96) in (94) we get

$$\begin{aligned} zQ(0,z) &= \int_0^\infty Q(0,z)e^{(-\lambda(1-z)x - N(x))}h(x)dx + \lambda z(z-1)p_0 \\ &= Q(0,z)\int_0^\infty e^{-\lambda(1-z)x}\big[e^{-N(x)}h(x)\big]dx + \lambda z(z-1)p_0 \\ &= Q(0,z)G^*(\lambda(1-z)) + \lambda z(z-1)p_0. \end{aligned}$$

Thus

$$Q(0,z) = \frac{\lambda z(z-1)p_0}{z - G^*(\lambda - \lambda z)}. \tag{97}$$

Now from (96) we have

$$\begin{aligned} Q(z) &= \int_0^\infty Q(x,z)dx \\ &= \int_0^\infty Q(0,z)e^{(-\lambda(1-z)x - N(x))}dx \\ &= Q(0,z)\int_0^\infty e^{(-\lambda(1-z)x}e^{-N(x)}dx \\ &= \frac{Q(0,z)}{\lambda(1-z)}\Big[1 - \int_0^\infty e^{-\lambda(1-z)x}\big(e^{-N(x)}h(x)\big)dx\Big] \end{aligned}$$

$$Q(z) = \frac{Q(0,z)}{\lambda(1-z)}[1 - G^*(\lambda - \lambda z)] \tag{98}$$

From (97) and (98) we get

$$Q(z) = \frac{z[G^*(\lambda-\lambda z)-1]p_0}{z-G^*(\lambda-\lambda z)} \tag{99}$$

Using L'Hospital rule, we get

$$\begin{aligned} Q(1) &= \lim_{z\to 1} Q(z) \\ &= p_0 \frac{[G^*(\lambda-\lambda z)-1]+z\lambda G^{*(1)}(\lambda-\lambda z)}{1+\lambda G^{*(1)}(\lambda-\lambda z)} \\ &= p_0 \frac{\lambda E(V)}{1-\lambda E(V)} \end{aligned}$$

From (95) we obtain

$$p_0 = 1 - \lambda E(V).$$

Hence

$$Q(z) = \frac{z[G^*(\lambda-\lambda z)-1][1-\lambda E(V)]}{z-G^*(\lambda-\lambda z)}. \tag{100}$$

7.2 Busy period

Let T be the length of a busy period (starting with a customer arrival to an idle server, until the becomes idle again). Define $B(t) = P(T \le t)$. Then $B(t)$ satisfies the relation

$$B(t) = \int_0^t \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} B^{*k}(t-u) dG(u) \tag{101}$$

The Laplace Stieltjes Transform (LST) of busy period $B(t)$ be denoted by $B^*(s)$. That is,

$$\begin{aligned} B^*(s) &= \int_0^{\infty} e^{-st} dB(t) \qquad (for Re(s) > 0) \\ &= \int_0^{\infty} e^{-st} \int_0^t \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} B^{*k}(t-u) dG(u) dt \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-su} \int_u^{\infty} e^{-s(t-u)} B^{*k}(t-u) dt dG(u) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-su} (B^*(s))^k dG(u) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda B^*(s) u)^k}{k!} e^{-(\lambda+s)u} dG(u) \\ &= \int_0^{\infty} e^{-(\lambda+s-\lambda B^*(s))u} dG(u) \end{aligned}$$

Therefore

$$B^*(s) = G^*(\lambda + s - \lambda B^*(s)). \tag{102}$$

From this the mean and higher moments of the number of customers in the system can be computed.

Conclusion:

We examined a queueing model offering n distinct services to which arrival is according to a MAP forming a single line. Service time has phase type distribution. A single server serves the customers. The service station provides two types of services - one is desirable and other is unwanted for each customer. If the service starts in an undesirable state then a clock also simultaneously starts ticking. In case this clock realizes before the exact requirement of the server is realized, then that customer leaves the system forever without being eligible for the service that he actually requires. On the other extreme, in case the correct identification of required service occurs before realization of clock, then the customer is served in that state and then leaves the system. In case right at the beginning of service the exact requirement of service is identified, then the customer starts getting that service right at the time when taken for service. Several system performance measures are evaluated. Applications of the model in hospital, telecommunication etc are indicated. Stochastic decomposition of the system state is analyzed. Some particular cases are indicated.

In a future work we propose to extend the model to multi server case.

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