AN ACCURACY OF ASYMPTOTIC FORMULAS IN CALCULATIONS OF A RANDOM NETWORK RELIABILITY

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INTRODUCTION

In this paper a problem of asymptotic and numerical estimates of relative errors for different asymptotic formulas in the reliability theory are considered. These asymptotic formulas for random networks are similar to calculations of Feynman integrals.

A special interest has analytic and numerical comparison of asymptotic formulas for the most spread Weibull and Gompertz distributions in life time models. In the last case it is shown that an accuracy of asymptotic formulas is much higher.

1. AN ASYMPTOTIC ESTIMATE OF A RELATIVE ERROR IN A DEFINITION OF A RELIABILITY LOGARITHM

Consider the nonoriented graph $\Gamma$ with fixed initial and final nodes and with the arcs set $W$. Define $R = \{R_1, ..., R_n\}$ as the set of all acyclic ways between the initial and final nodes of the graph $\Gamma$. Designate $P_R$ the probability of the way $R$ work. Then in the condition

$$p_w \sim \exp(-c_w h^{-d_w}), \ h \to 0, \ w \in W,$$

we have:

$$P_R \sim \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_{R}}(1 + o(1))\right),$$

where $C(R) = \sum_{w,d_w=D(R)} c_w$ and $D'_R < D_R$ is a next by a quantity after $D_R = \max_{weR} d_w$ element in the set $\{d_w, w \in R\}$, $C'(R) = \sum_{w,d_w=D(R)} c_w$. If in the way $R$ this element is absent we put then $D'_R = -\infty$,

$C'(R) = 0$.

Denote $D_\Gamma = \min_{R \in R} D_R$ and designate $R_4 = \{R : D_R = D_\Gamma\}$, $R_2 = R \setminus R_4$, then the probability $P_{R_1}$ of the graph $\Gamma$ work satisfies the formulas

$$P_{R_1} \sim P_{R_1}^1 + P_{R_1}^2, \ P_{R_1}^i = \sum_{R \in R_i} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_{R}}(1 + o(1))\right), \ i = 1, 2.$$

By the definition

$$P_{R_1}^1 \sim \sum_{R \in R_4} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D'_{R}}(1 + o(1))\right) \sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \sum_{R \in R_4} \exp\left(-C'(R)h^{-D'_{R}}(1 + o(1))\right),$$

where $C_\Gamma = \min_{R \in R_4} C_R$ and $D'_R < D_\Gamma$, $R \in R_4$, so

$$P_{R_1}^1 \sim \exp\left(-C_\Gamma h^{-D_\Gamma}\right) \exp\left(-C'_\Gamma h^{-D'_\Gamma}(1 + o(1))\right),$$

where $D'_\Gamma = \min_{R \in R_4, C(R) = C_\Gamma} D'_R < D_\Gamma$, $C'_\Gamma = \min_{R \in R_4, C(R) = C_\Gamma} C'(R)$. 

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And consequently $D_R < D^*_R$, $R \in R_2$, $P^2_R = o(P^1_R)$:

$$P^2_R \sim \sum_{R \in R_2} \exp\left(-C(R)h^{-D_R} - C'(R)h^{-D_R} (1 + o(1))\right) \sim \sum_{R \in R_2} \exp\left(-C(R)h^{-D_R} (1 + o(1))\right) =$$

$$= \exp\left(-C_1 h^{-D_1}\right) \sum_{R \in R_2} \exp\left(-C_1 h^{-D_1} - C(R)h^{-D_R} (1 + o(1))\right) \sim$$

$$\sim \exp\left(-C_1 h^{-D_1}\right) \sum_{R \in R_2} \exp\left(-C(R)h^{-D_R} (1 + o(1))\right) \sim$$

$$\sim \exp\left(-C_1 h^{-D_1}\right) \exp\left(-C^*_1 h^{-D^*_1} (1 + o(1))\right),$$

where

$$D^*_R = \min_{R \in R_2} D_R > D^*_1, \quad C^*_R = \min_{R \in R_2} C(R).$$

So we have:

$$P^*_R \sim \exp\left(-C_1 h^{-D_1}\right) \left(\exp\left(-C^*_1 h^{-D^*_1} (1 + o(1))\right) + \exp\left(-C^*_1 h^{-D^*_1} (1 + o(1))\right)\right) \sim$$

$$\sim \exp\left(-C_1 h^{-D_1}\right) \exp\left(-C^*_1 h^{-D^*_1} (1 + o(1))\right).$$

As a result obtain that

$$\ln P^*_R \sim -C_1 h^{-D_1} \left(1 + Ah^{\Delta_1} (1 + o(1))\right), \quad \Delta_1 = D^*_1 - D_1 > 0, \quad A = C^*_1 / C_1.$$

And consequently

$$\frac{\ln P^*_R}{-C(\Gamma)h^{-D^*_1}} - 1 \sim Ah^{\Delta_1}. \quad (1)$$

2. AN ASYMPTOTIC ESTIMATE OF A RELATIVE ERROR IN A DEFINITION OF A RELIABILITY

Assume that $P(U_p)$ is the probability of the event $U_p$ that all arcs $w^p_1,...,w^p_{m_p}$ of the way $R_p$ work. Then we have

$$P^*_p = P\left(\bigcup_{p=1}^n U_p\right). \quad (2)$$

Suppose that the probability of the arc $w \in W$ work equals $\exp\left(-c_w h^{-d_w}\right), \quad h > 0$, where $c_w,d_w$ are some positive numbers and for arcs $w' \neq w^*$ the constants $d_{w'} \neq d_{w^*}$. So we have

$$P(U_p) = \exp\left(-\sum_{j=1}^{m_p} c_{w^j} h^{-d_{w^j}}\right).$$

Assume that the enumeration of the arcs in the way $R_p$ satisfies the inequalities

$$d_{w^*_1} > d_{w^*_2} > ... > d_{w^*_{m_p}}.$$
Denote \( D^p = \{d_{w^p_1}, ..., d_{w^p_{mp}}\} \) and introduce on the vectors set \( \{D^p, 1 \leq p \leq n\} \) the following order relation. Say that \( D^p \succ D^q \), if for some \( k \leq \min\{m_p, m_q\} \) the first \( k - 1 \) components of these vectors coincide and the \( k \) component in the vector \( D^p \) is larger than in the vector \( D^q \). If there is not such \( k \) and in the vectors \( D^p, D^q \) all first \( \min\{m_p, m_q\} \) components coincide then \( D^p \succ D^q \) for \( m_p < m_q \).

Remark that for some \( p \neq q \) the arcs sets \( \{w \in R_p\}, \{w \in R_q\} \) can not satisfy the inclusion \( \{w \in R_p\} \subseteq \{w \in R_q\} \). In the opposite case there is the node \( u_* \) in which the ways \( R_p, R_q \) diverge by the arcs \( (u_*, u_p) \) \( (u_*, u_q) \). But as the arc \( (u_*, u_p) \in \{w \in R_q\} \) then the way \( R_q \) has a cycle. This conclusion contradicts with the assumption that the way \( R_q \) is acyclic.

So as the quantities \( d_w \) are different then \( D^p \neq D^q, p \neq q \). As a result we obtain the order relation on the vectors set \( \{D^1, ..., D^n\} \), and if \( D^p \succ D^q, h \to 0 \), so \( P(U_q) = o(P(U_p)) \). It is not difficult to check that this relation is transitive. Consequently the order relation on the set \( \{D^1, ..., D^n\} \) is linear. Assume that the enumeration of the vectors \( D^p \) satisfies the formula \( D^1 \succ ... \succ D^n \). From the formula (2) we have

\[
R^*_p - \sum_{1 \leq p < q \leq m} P(U_p U_q) \leq P_t \leq R^*_p, R^*_p = \sum_{p=1}^{m} P(U_p).
\] (3)

As the inclusion \( \{w \in R_p\} \subseteq \{w \in R_q\} \) is not true for \( p \neq q \) so in the way \( R_p \) there is an arc which does not belong to the way \( R_q \). Consequently we have

\[
P(U_q) = o\left(P(U_p)\right), 1 \leq p < q \leq m, \sum_{1 \leq i < j \leq m} P(U_p U_q) = o(P(U_2))
\] (4)

The formulas (3), (4) give us the following asymptotic expansion for \( P_t \) with the first and the second members of the smallness:

\[
P_t \sim R^*_p \sim P(U_1), P_t - P(U_1) \sim P(U_2), P(U_2) = o(P(U_1)), h \to 0.
\] (5)

3. AN APPLICATION TO LIFE TIME MODELS

Suppose that \( \tau_w \) are independent random variables and characterize life times of the arcs \( w \in W \). Denote \( p_w(h) = P(\tau_w > t) \) and designate the life time of the graph \( \Gamma \) by

\[
\tau_\Gamma = \min_{R \in K} \max_{w \in K} \tau_w.
\]

If \( h = 1/t \) then we have with \( t \to \infty \) the Weibull distributions of the arcs life times and the formula

\[
\frac{\ln P(\tau_\Gamma > t)}{-C(\Gamma)t^{P_\Gamma}} - 1 = g(t) \sim \frac{A}{t^{*-\lambda t}} = G(t)
\] (6)

If \( h = \exp(-t) \), \( t \to \infty \), then we have the Gompertz distributions of the arcs life times and the formula (1) transforms into

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\[
\frac{\ln P(\tau_\Gamma > t)}{-C(\Gamma) \exp(D_\Gamma t)} - 1 = g_1(t) \sim G_1(t) = G(\exp(t)),
\]
so \( G_1(t) = o(G(t)) \).

Consequently for the Gompertz distributions the convergence rate in the asymptotic (7) is much faster than for the Weibull distributions in (6).

If \( h = 1/t, \ t \to \infty \), then for the Weibull distributions of the arcs life times the formula (5) transforms into

\[
P(\tau_\Gamma > t) \sim \exp\left(-\sum_{j=1}^{m} c_{w_j} t^{d_{w_j}}\right),
\]

\[
\frac{P(\tau_\Gamma > t)}{\exp\left(-\sum_{j=1}^{m} c_{w_j} t^{d_{w_j}}\right)} - 1 = f(t) - F(t) = o(1), \ F(t) = \exp\left(-\sum_{j=1}^{m} c_{w_j} t^{d_{w_j}} + \sum_{j=1}^{m} c_{w_j} t^{d_{w_j}}\right).
\]

If \( h = \exp(-t), \ t \to \infty \), then for the Gompertz distributions of the arcs life times the formula (5) transforms into

\[
P(\tau_\Gamma > t) \sim \exp\left(-\sum_{j=1}^{m} c_{w_j} \exp(d_{w_j} t)\right), \quad \frac{P(\tau_\Gamma > t)}{\exp\left(-\sum_{j=1}^{m} c_{w_j} \exp(d_{w_j} t)\right)} - 1 = f_1(t) \sim F_1(t) = F(\exp(t)),
\]
so \( F_1(t) = o(F(t)) \).

Consequently for the Gompertz distributions the convergence rate in the asymptotic (9) is much faster than for the Weibull distributions in (8).

For \( h = 1/t \) denote \( \left| P_1^\prime - P_1 \right| / P_1 = A(t) \), and for \( h = \exp(-t) \) designate \( \left| P_1^\prime - P_1 \right| / P_1 = A_1(t) \). It is clear that \( A_1(t) = A(\exp(t)) \) tends to zero for \( t \to \infty \) much faster than \( A(t) \).

From this section we see that the Gompertz distributions of the arcs life times (these distributions are preferable in life time models of alive [1] and of complex information [2] systems), give much more accuracy asymptotic formulas than the Weibull distributions. These both distributions are limit for a scheme of a minimum of independent and identically distributed random variables.

4. RESULTS OF NUMERICAL EXPERIMENTS FOR BRIDGE SCHEMES

![Fig.1 The bridge scheme.](image)
Consider the bridge scheme $\Gamma$ represented on the Fig. 1 with the parameters $d_1 = 0.02$, $d_2 = 0.09$, $d_3 = 0.5$, $d_4 = 0.72$, $d_5 = 0.2$. Calculate the functions $f(t), f_1(t), A(t), A_1(t), g(t), g_1(t)$. 

Fig.2 The relative errors $f(t)$ and $f_1(t)$ in the reliability $P_\Gamma$ calculations

Fig.3 The relative errors $A(t)$ and $A_1(t)$ in the reliability $P_\Gamma$ calculations

Fig.4 The relative errors $g(t)$ and $g_1(t)$ in $\ln P_\Gamma$ calculations.

The results of the numerical experiments represented above show that a transition from the Weibull to the Gompertz distribution decreases significantly relative errors in calculations of the
reliability and its logarithm. The asymptotic estimate $R_1^*$ of the reliability $R_1$ is better than $P(U_1)$. The relative error of the $\ln R_1$ calculation is larger than the relative error of the $R_1$ calculation. But a complexity of the $\ln R_1$ calculation is smaller.

REFERENCES