BOTTLENECKS IN GENERAL TYPE LOGICAL SYSTEMS WITH UNRELIABLE ELEMENTS

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In this paper a model of general type logical system with unreliable elements [1], [2] is considered. An asymptotic analysis of its work (failure) probability is made in appropriate conditions on work (failure) probabilities of the system elements. A concept of bottlenecks of this system is constructed on a suggestion that an increase (a decrease) of elements reliabilities lead to an increase (a decrease) of the system reliability.

A construction of general type logical system is founded on concepts of disjunctive and conjunctive normal forms (DNF and CNF) of a logical function. This approach allows obtaining main results in maximal general and convenient for engineering calculations form comparatively with recursive definitions of logical functions used in [3].

Denote $Z$ the set which consists of $|Z|$ independent random logical variables $z$, $I \subseteq \{1, 2, \ldots, 2^{|Z|}\}$. Consider the logical function $A$ represented in DNF

$$A = \bigvee_{i \in I} \left[ \bigwedge_{z \in Z_i} z \bigwedge \bigwedge_{\bar{z} \in \overline{Z_i}} \bar{z} \right].$$

Here the family $\left\{ (Z_i, \overline{Z_i}) : i \in I \right\}$ consists of the sets pairs $Z_i, \overline{Z_i} \subseteq Z$, $Z_i \cap \overline{Z_i} = \emptyset$, and for $i \neq j (Z_i, \overline{Z_i}) \neq (Z_j, \overline{Z_j})$. Suppose that $p_z = P(z = 1)$, $q_z = P(z = 0)$, $p_z + q_z = 1$, and random variables $z \in Z$ are independent. The logical function $A$ with random arguments $z \in Z$ is denoted by $A$ and called the logical system.

Low reliable elements

Suppose that for $\forall z \in Z$

$$\exists c(z), c(z) > 0 : p_z = p_z(h) \sim \exp\left(-h^{-c(z)}\right), h \to 0.$$  \hspace{1cm} (2)

Denote $C = \min_{i \in I} \max_{z \in Z_i} c(z)$,

$$I' = \left\{ i \in I : \max_{z \in Z_i} c(z) = C \right\},$$

$$S_i = \left\{ z \in Z_i : c(z) = C \right\},$$

$$S_i \subseteq S_i, i \in I',$$

$$\mathcal{S} = \left\{ S_i : i \in I' \right\},$$

$$N(\mathcal{S}) = \min \left( |S| : S_i \in \mathcal{S} \right),$$

$$\mathcal{T} = \left\{ \left\{ z_i \in S_i, i \in I' \right\} : \left\{ z_i \in S_i, i \in I' \right\} \right\},$$

$$N(\mathcal{T}) = \min \left( |T| : T \in \mathcal{T} \right).$$

and let $\mathcal{S}', \mathcal{T}'$ are families of minimal by an inclusion sets from the families $\mathcal{S}, \mathcal{T}$,
\[ S' = \{ S \in S': |S| = N(S) \}, \ T' = \{ T \in T': |T| = N(T) \}. \]

**Theorem 1.** If the formulas (1), (2) are true then

\[ -\ln P(A = 1) \sim N(S) h^{-C}, \ h \to 0. \]  \hspace{1cm} (3)

**Proof.** Rewrite the logical function \( A \) as follows

\[ A = \bigvee_{i \in I} \left[ \bigwedge_{z \in Z_i} z \bigwedge A_i \right], \ A_j = \bigvee_{k \in J_i} \left( \bigwedge_{z \in Z_j} z \right), \ J_i = \{ k : Z_k = Z_i \}. \]

The formula (2) leads to \( p_z = P(z = 1) \to 0, \ h \to 0 \), so

\[ P(A = 0) = \prod_{z \in Z_i, Z_i = Z_j} p_z \to 0, \ h \to 0. \]

If the obvious that

\[ \sum_{i \in I} \prod_{z \in Z_i} p_z P(A_i = 1) - \sum_{i, j \in I, i \neq j} P\left( A_i \prod_{z \in Z_i} z = 1 \bigcap A_j \prod_{z \in Z_j} z = 1 \right) \leq P(A = 1) \leq \prod_{i \in I} \prod_{z \in Z_i} p_z, \]  \hspace{1cm} (4)

As for \( i \neq j \)

\[ P(A_i A_j = 1) = P\left( \sum_{k \in J_i, k \neq J_j} \prod_{z \in Z_{i,j}} z = 1 \bigcap \prod_{z \in Z_{i,j}} z = 1 \right) \geq \prod_{i \in I} q_z \to 1, \ h \to 0, \]

and

\[ \sum_{i, j \in I, i \neq j} P\left( A_i \prod_{z \in Z_i} z = 1 \bigcap A_j \prod_{z \in Z_j} z = 1 \right) = P(A_i A_j = 1) \prod_{z \in Z_i \cup Z_j} p_z, \]

So from the formula (4) obtain

\[ P(A = 1) = \sum_{i \in I} \prod_{z \in Z_i} p_z \sim \sum_{i \in I} \exp\left( -\sum_{z \in Z_i} \ell(z) \right), \ h \to 0. \]  \hspace{1cm} (5)

Denote \( C_i = \max_{z \in Z_i} c(z), \ K_i = \{ z \in Z_i : c(z) = C_i \} \). The formulas

\[ \sum_{z \in Z_i} h^{-\ell(z)} \sim h^{-C_i} |K_i|, \ h \to 0, \]

and (5) give

\[ P(A = 1) \sim \sum_{i \in I} \exp\left( -h^{-C_i} (1 + o(1)) |K_i| \right), \ h \to 0. \]

Consequently,

\[ P(A = 1) = \sum_{i \in I'} \exp\left( -h^{-C_i} (1 + o(1)) |S_i| \right) \sim \sum_{i \in I'} \exp\left( -h^{-C_i} (1 + o(1)) |S_i| \right) = \exp\left( -h^{-C} (1 + o(1)) N(S) \right)[i \in I': |S_i| = N(S) |]. \]
So formula (3) is true.

**Remark 1.** Suppose that \( \tau(z) \) are independent random variables equal to life times of logical elements \( z \), and \( h = h(t) \) - is monotonically decreasing and continuous function, \( h \to 0, \ t \to \infty \). Then the asymptotic

\[
P(\tau(z) > t) = p_z(h) \sim \exp\left(-h^{-c(z)}\right), \ t \to \infty.
\]

Is character for the Weibull distribution which is widely used in life time models of complex systems with old and so low reliable elements [4], [5].

**Highly reliable elements**

Suppose that for \( \forall z \in Z \)

\[
\exists c(z), \ c(z) > 0: q_z(h) = q_z(h) \sim \exp\left(-h^{-c(z)}\right), \ h \to 0
\]

Consider the logical function \( A \) represented in CNF

\[
A = \bigwedge_{i \in I} \left[ \bigvee_{z \in Z_i} z \bigvee_{z \in Z_i} \bar{z} \right].
\]

**Theorem 2.** If the formulas (6), (7) are true then

\[
- \ln P(A = 0) \sim N(s) h^{-C}, \ h \to 0.
\]

**Remark 2.** Suppose that \( \tau(z) \) are independent random variables equal to life times of logical elements \( z \), \( u \ h = h(t) \) - is monotonically increasing and continuous function, \( h \to 0, \ t \to 0 \). Then the asymptotic

\[
P(\tau(z) \leq t) = q_z(h) \sim \exp\left(-h(t)^{-c(z)}\right), \ t \to 0
\]

Is character for the Weibull distribution which is widely used in life time models of complex systems with young and so high reliable elements.

**Mixing case**
Suppose that the sets $X_i, V_i, \overline{X}_i, \overline{V}_i \subseteq Z$ are nonintersecting. For $\forall z \in X_i \cup \overline{X}_i$ the formula (2) is true and for $\forall z \in V_i \cup \overline{V}_i$ the formula (6) taking place, $i \in I$. So low reliable and high reliable elements in the system $A$ are present simultaneously

**Theorem 3.** Suppose that

$$A = \bigvee_{i \in I} \left[ \bigwedge_{z \in X_i \cup V_i} z \bigwedge_{z \in X_i \cup \overline{V}_i} \overline{z} \right]$$

(9)

Then for $Z_i = X_i \cup \overline{V}_i \neq \emptyset, Z_i = V_i \cup \overline{X}_i, i \in I$, the formula (3) is true.

Suppose that

$$A = \bigwedge_{i \in I} \left[ \bigvee_{z \in X_i \cup V_i} z \bigvee_{z \in X_i \cup \overline{V}_i} \overline{z} \right].$$

Then for $Z_i = V_i \cup \overline{X}_i \neq \emptyset, Z_i = X_i \cup \overline{V}_i, i \in I$, the formula (8) is true.

**Concept of bottlenecks**

Define bottlenecks in logical system $A$

**Theorem 4.** Suppose that $\varepsilon_0 = \min \left( \left| C - c(z) \right| > 0 : z \in Z \right)$.

1. For any $S_i \in S$ and each $\varepsilon, 0 < \varepsilon < \varepsilon_0$, the property $(B)$ is true: the replacement $c(z)$ by $c(z) - \varepsilon$ for all $z \in S_i$ leads to the replacement $C \rightarrow C - \varepsilon$.

2. If a set $S \subseteq Z$ and satisfies the condition $(B)$, then $S_0 \in S : S_0 \subseteq S$.

3. For any $T \in T$ and each $\varepsilon, 0 < \varepsilon < \varepsilon_0$, the property $(C)$ is true: the replacement $c(z)$ by $c(z) + \varepsilon$ for all $z \in T$ leads to the replacement $C \rightarrow C + \varepsilon$.

4. If a set $T \subseteq Z$ and satisfies the condition $(C)$, then $\exists T_0 \in T : T_0 \subseteq T$.

**Proof.** Proof the statements 1, 3, as the statements 2, 4 are trivial.

1. If $c(z)$ is replace by $c(z) - \varepsilon$, $z \in S_i$, then

$$\max_{z \in Z_i} c(z) = C - \varepsilon, \max_{z \in Z_i} c(z) \geq C - \varepsilon, j \neq i \Rightarrow \min_{i \in I} \max_{z \in Z_i} c(z) = C - \varepsilon.$$  

2. If $c(z)$ is replace by $c(z) + \varepsilon$, $z \in T$, then

$$\max_{z \in Z_i} c(z) = C + \varepsilon, i \in I', \max_{z \in Z_i} c(z) \geq C + \varepsilon, j \neq I' \Rightarrow \min_{i \in I} \max_{z \in Z_i} c(z) = C + \varepsilon.$$  

**Corollary 1.** The statements 2, 4 of the theorem 4 establish that the families $S', S', T', T''$ and the numbers $C, N(S), N(T)$ do not depend on a view of DNF (of KNF) of the logical function $A$. 
Proof. Suppose that the theorem 1 condition is true, all other case is considered analogically. Denote by $A_1, A_2$ - DNF, which define the logical function $A$, $S_1$, $S_2$ are families of subsets $Z$, created by $A_1$, $A_2$, and $S'_1$, $S'_2$ are families of minimal sets from the families $S_1$, $S_2$, correspondingly. If the set $S_1 \in S'_1$ then it satisfies the property (B) and so $\exists S_2 \in S'_2 : S_2 \subseteq S_1$. Analogously if $S_2 \in S'_2$ then there is $S_1^* \in S'_1 : S_1^* \subseteq S_2$. Consequently $S_1^* \subseteq S_2 \subseteq S_1$ and the families $S'_1$, $S'_2$ definition leads to the equality $S_1^* = S_2 = S_1$ and so $S'_1 = S'_2$. Thus, the family $S'$ does not depend on a view of logical function $A$ DNF. Similar statements may be proved for the families $T', T'', S'$. For the numbers $N(T), C, N(S)$ the statements of the corollary 1 may be obtain from the formula (3).

Remark 3. The statements 1 (the statements 3) of the theorem 4 establishes that an increase of elements $z \in S$ reliabilities for any set $S \in S$ (a decrease of elements $z \in T$ reliabilities for any set $T \in T$) leads to an increase (to a decrease) of system $A$ reliability. The corollary 1 allows to call sets from the families $S', S'', T', T''$ by bottlenecks in logical system $A$.

Remark 4. Suppose that $\forall z \in Z$ the condition (2) or the condition (6) are replaced by

$$\exists c(z), \ d(z), \ c(z) > 0, \ d(z) > 0 : p_z = p_z(h) \sim \exp\left(-d(z)h^{-c(z)}\right), \ h \to 0,$$

or by

$$\exists c(z), \ d(z), \ c(z) > 0, \ d(z) > 0 : q_z = q_z(h) \sim \exp\left(-d(z)h^{-c(z)}\right), \ h \to 0,$$

correspondingly. Then to obtain the formula (3) or the formula (8) correspondingly it is enough to redefine $|S|, S \subseteq Z$, and put (besides of number of elements in a set $S$):

$$|S| = \sum_{z \in S} d(z).$$

References


